College Algebra M121 Notes Sections 3-4 → 3-6

Section 3.4
Zeros of Polynomials

Do I Have to Check My Bag?

Airlines have regulations on the sizes of carry-on luggage that are allowed. As a passenger, you are interested in the volume of your luggage, but the airline is concerned about the sum of bag's length, width, and depth.
In this section's Exercise Set, you will work with a polynomial function that relates the two quantities and allows you to find dimensions of a carry-on bag that meet both your volume requirement and the airline's regulations.

Objective #1: Use the Rational Zero Theorem to find possible rational zeros.

✔ *Solved Problem #1*

1. List all possible rational zeros of
$f(x) = 4x^5 + 12x^4 - x - 3$.

Factors of the constant term -3: $\pm1, \pm3$
Factors of the leading coefficient 4: $\pm1, \pm2, \pm4$

The possible rational zeros are:

$$\frac{\text{Factors of } -3}{\text{Factors of } 4} = \frac{\pm1, \pm3}{\pm1, \pm2, \pm4}$$
$$= \pm1, \pm3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}$$

Pencil Problem #1

1. List all possible rational zeros of p
$f(x) = 3x^4 - 11x^3 - x^2 + 19x + 6$.
q

$$\frac{\pm1 \pm2 \pm3 \pm6}{\pm1 \pm3}$$

$\pm1 \pm2$
$\pm3 \pm6$
$\pm\frac{1}{3} \pm\frac{2}{3}$

Pool of possible Rational Roots
nice

Objective #2: Find zeros of a polynomial function.

✔ *Solved Problem #2*

2. Find all zeros of $f(x) = x^3 + x^2 - 5x - 2$.

First, list the possible rational zeros:

$$\frac{\text{Factors of } -2}{\text{Factors of } 1} = \frac{\pm1, \pm2}{\pm1} = \pm1, \pm2$$

Now use synthetic division to find a rational zero from among the list of possible rational zeros. Try 2:

$$\begin{array}{r|rrrr} 2 & 1 & 1 & -5 & -2 \\ & & 2 & 6 & 2 \\ \hline & 1 & 3 & 1 & 0 \end{array}$$

The last number in the bottom row is 0.
Thus 2 is a zero and $x - 2$ is a factor.

The first three numbers in the bottom row of the synthetic division give the coefficients of the other factor. This factor is $x^2 + 3x + 1$.

Pencil Problem #2

2. Find all zeros of $f(x) = 1x^3 + 4x^2 - 3x - 6$.
q p

$$\frac{\pm1 \pm2 \pm3 \pm6}{\pm1}$$

$\pm1 \pm2$
$\pm3 \pm6$

Possible pool

Now put in Calculator graph and See which one out of pool it looks like is a root... then Synthetic divide to find other factor

looks like $x = -1$ is a root so $x+1$ would be a factor lets check

$$\begin{array}{r|rrrr} -1 & 1 & 4 & -3 & -6 \\ & \downarrow & -1 & -3 & 6 \\ \hline & 1 & 3 & -6 & 0 \end{array}$$

Whoooo it is a root

over →

Factor completely: $x^3 + x^2 - 5x - 2 = 0$

$$(x-2)(x^2+3x+1)=0$$

Since x^2+3x+1 is not factorable, use the quadratic formula to find the remaining zeros.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)} = \frac{-3 \pm \sqrt{5}}{2}$$

The zeros are 2 and $\frac{-3 \pm \sqrt{5}}{2}$.

So we now know $(x-1)$ and x^2+3x-6 are factors

now set $=0$ and solve

$(x-1)(x^2+3x-6)=0$

Must use quadratic formula

$x=1$

$x = \frac{-3 \pm \sqrt{33}}{2}$

$x = \frac{-3 \pm \sqrt{(3)^2-4(-1)(-6)}}{2(1)}$

$x = \frac{-3 \pm \sqrt{9+24}}{2} = x = \frac{-3 \pm \sqrt{33}}{2}$

Objective #3: Solve polynomial equations.

✔ Solved Problem #3

3. Solve: $x^4 - 6x^3 + 22x^2 - 30x + 13 = 0$

First, list the possible rational roots:

$$\frac{\text{Factors of } 13}{\text{Factors of } 1} = \frac{\pm 1, \ \pm 13}{\pm 1} = \pm 1, \ \pm 13$$

Now use synthetic division to find a rational root from among the list of possible rational roots. Try 1.

$$\begin{array}{r|rrrrr} 1 & 1 & -6 & 22 & -30 & 13 \\ & & 1 & -5 & 17 & -13 \\ \hline & 1 & -5 & 17 & -13 & 0 \end{array}$$

The last number in the bottom row is 0.
Thus, 1 is a root.

Rewrite the equation in factored form using the bottom row of the synthetic division to obtain the coefficients of the other factor.

$$x^4 - 6x^3 + 22x^2 - 30x + 13 = 0$$

$$(x-1)(x^3 - 5x^2 + 17x - 13) = 0$$

Use the same approach to find another root. Try 1 again.

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 17 & -13 \\ & & 1 & -4 & 13 \\ \hline & 1 & -4 & 13 & 0 \end{array}$$

The last number in the bottom row is 0.
Thus, 1 is a root (of multiplicity 2).

The first three numbers in the bottom row of the synthetic division give the coefficients of the factor $x^2 - 4x + 13$.

✎ Pencil Problem #3 ✎

3. Solve: $1x^3 - 2x^2 - 11x + 12 = 0$

Pool of Poss

$\pm 1 \ \pm 2 \ \pm 3 \ \pm 4 \ \pm 6 \ \pm 12$

$$\frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1}$$

Now graph on Calc. to find rational root Must be one of those in Pool. If more than one it doesn't Matter just pick one to Synt. divide

Looks like 1 is a root so $x-1$ would be a factor

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -11 & 12 \\ & & 1 & -1 & -12 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

Whoooohoo yes it is a factor

Now write in factored form Set $=0$ Solve

$(x-1)(x^2-x-12)=0$

factor Completley

$(x-1)(x-4)(x+3)$ $\quad -12$, -1, $-4, 3$

$x=1$ $\quad x=4$ $\quad x=-3$

3 real roots all Nice!! :)

Since $x^2 - 4x + 13$ is not factorable, use the quadratic formula to find the remaining roots.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{-36}}{2}$$

$$x = \frac{4 \pm 6i}{2}$$

$$x = 2 \pm 3i$$

The roots are 1 and $2 \pm 3i$.

Objective #4: Use the Linear Factorization Theorem to find polynomials with given zeros.

✔ Solved Problem #4

4. Find a third-degree polynomial function $f(x)$ with real coefficients that has -3 and i as zeros such that $f(1) = 8$.

Because i is a zero and the polynomial has real coefficients, the conjugate, $-i$, must also be a zero. We can now use the Linear Factorization Theorem.

$$\begin{aligned} f(x) &= a_n(x - c_1)(x - c_2)(x - c_3) \\ &= a_n(x - (-3))(x - i)(x - (-i)) \\ &= a_n(x+3)(x - i)(x+i) \\ &= a_n(x+3)(x^2 - i^2) \\ &= a_n(x+3)(x^2 - (-1)) \\ &= a_n(x+3)(x^2+1) \\ &= a_n(x^3+3x^2+x+3) \end{aligned}$$

Now we use $f(1) = 8$ to find a_n.

$$f(1) = a_n(1^3 + 3\cdot 1^2 + 1 + 3) = 8$$
$$8a_n = 8$$
$$a_n = 1$$

Now substitute 1 for a_n in the formula for $f(x)$.

$f(x) = 1(x^3+3x^2+x+3)$

or $f(x) = x^3+3x^2+x+3$

✎ Pencil Problem #4 ✎

4. Find a fourth-degree polynomial function $f(x)$ with real coefficients that has i and $3i$ as zeros such that $f(-1) = 20$.

degree of 4 you know

i is a root so then $-i$

$3i$ " " " so then $-3i$

$f(x) = a_n(x-i)(x+i)(x+3i)(x-3i)$

all 4 roots accounted for all imaginary

Now foil out

$(x^2 \cancel{-xi} \cancel{+xi} - i^2)(x^2 + 3xi - 3xi - 9i^2)$

$+1$ $\quad$ $+9$

$(x^2+1)(x^2+9)$

$x^4 + 9x^2 + 1x^2 + 9$

$f(x) = a_n(x^4 + 10x^2 + 9)$ We were given

$f(-1) = 20$ So plug in -1 for x and 20 for $f(x)$ and solve for a_n ← leading coeft.

$20 = a_n((1)^4 + 10(1)^2 + 9)$

$1+10+9$

$20 = a_n(20)$

$\frac{20}{20} = \frac{a_n(20)}{20}$

$1 = a_n$

answer → $\boxed{f(x) = 1(x^4 + 10x^2 + 9)}$

Objective #5: Use Descartes's Rule of Signs.

✔ *Solved Problem #5*

5. Determine the possible numbers of positive and negative real zeros of

$f(x) = x^4 - 14x^3 + 71x^2 - 154x + 120.$

+ - + - +

1 2 3 4

Count the number of sign changes in $f(x)$.

$f(x) = x^4 - 14x^3 + 71x^2 - 154x + 120$

Since $f(x)$ has four sign changes, it has 4, 2, or 0 positive real zeros.

now plug in -x

Count the number of sign changes in $f(-x)$.

$f(-x) = (-x)^4 - 14(-x)^3 + 71(-x)^2 - 154(-x) + 120$

+ + + + +

$= x^4 + 14x^3 + 71x^2 + 154x + 120$

Since $f(-x)$ has no sign changes, $f(x)$ has 0 negative real zeros.

all roots could be 4 Real 0 img, 2 Real 2 img, 0 Real 4 imag

hand game

Pencil Problem #5

5. Determine the possible numbers of positive and negative real zeros of $f(x) = x^3 + 2x^2 + 5x + 4$.

Positive roots check # of sign changes

$x^3 + 2x^2 + 5x + 4$

+ + + +

No sign changes

So there are no positive roots. So that means because degree is 3, that it crosses the x-axis at negative x-intercepts either 3 or 1 times (hand game)

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. $\pm1,\ \pm2,\ \pm3,\ \pm6,\ \pm\frac{1}{3},\ \pm\frac{2}{3}$ *(3.4 #3)*

2. $-1,\ \frac{-3-\sqrt{33}}{2}$, and $\frac{-3+\sqrt{33}}{2}$ *(3.4 #13)* typo look

3. $\{-3, 1, 4\}$ *(3.4 #17)*

4. $f(x) = x^4 + 10x^2 + 9$ *(3.4 #29)* Beautiful

5. f has no positive real zeros and either 3 or 1 negative real zeros *(3.4 #33)*